The (hopefully) Definitive Guide to Coordinate Systems and Transformation in the Low Energy Hall A Experiments

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1 Introduction

In the upcoming polarized beam Hall A experiment the polarization of a detected proton (or deuteron) is measured in the Focal Plane Polarimeter (FPP) installed in the detector stack of the left HRS. As the name implies the FPP measures the polarization components of the detected particle in the *focal plane* using the coordinate system of the FPP[1] (see 2.4). The physics observables, however are calculated by theorists in different coordinate system depending on the reaction under study. It is therefore necessary to perform a coordinate transformation on the extracted polarization in order to allow comparison with the theory (it should also be noted that since the definition of the FPP coordinate system varies per event, the calculation needs to be performed on an event by event basis, or at least for small bins, before the extraction of the polarization components).
The transformation from the frame in which the physics observables are reported (henceforth the "physics frame") and the FPP proceeds through the transport (HRS) frame and the spin transport matrix which rotates the spin of the detected particle on the magnetic field of the HRS.
In total the polarization of the detected particle in the FPP may be written as:

$$P_{FPP} = R_{fpp} R_{spin\,transport} R_{transport} P_{physics}$$

Where $P_{physics}$ is the polarization vector in the physics frame, $R_{transport}$ is the transformation matrix from the physics to the transport frame, $R_{spin\,transport}$ is the spin rotation matrix in the HRS, R_{fpp} is the transformation matrix from the spin rotated transport frame to the HRS frame and P_{FPP} is the polarization vector measured in the FPP. Note also that apart from the definition of the physics frame (and hence $R_{transport}$) the transformation does not depend on the reaction under study.
This document will detail the various coordinate systems and transformation used in the upcoming low energy experiments.

[1] For an overview of the FPP operation see [1, 2, 5, 6] and many other sources.

2 ep (and eD) Elastic Scattering

2.1 The Physics Coordinate System

In ep elastic scattering the polarization vector of the proton is described in the lab frame of the proton, the physics observables are: $\vec{k_i}$ the electron beam momentum, $\vec{k_f}$ the electron beam momentum, $\vec{q}$ the 3-momentum transfer (also equal by construction to the scattered proton momentum).
The coordinate system is defined by:

$$\begin{aligned}
\hat{z} =& \frac{\vec{k_i} - \vec{k_f}}{|\vec{k_i} - \vec{k_f}|} = \hat{q} = \hat{p} \\
\hat{y} =& \frac{\vec{k_i} \times \vec{k_f}}{|\vec{k_i} \times \vec{k_f}|} = \frac{\vec{q} \times \vec{k_i}}{|\vec{q} \times \vec{k_i}|} \\
\hat{x} =& \hat{y} \times \hat{z}
\end{aligned}$$

2.2 The Transport System and $R_{transport}$

The transport (HRSL) system is defined so that the $\hat{z}$ direction is along the central axis of the HRS, $\hat{x}$ points downwards (towards higher momentum particles in the magnetic field) and $\hat{y} = \hat{z} \times \hat{x}$ (towards beam left).
The rotation matrix from the scattering frame to the transport frame is given by the column vectors ($\hat{x}$,$\hat{y}$,$\hat{z}$), where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are expressed in the fixed lab frame. $\vec{k_i}$ is along the z axis of the lab frame, so in the transport frame:

$$\vec{k_i} = \begin{pmatrix} 0 \\ -sin\theta_{hrs} \\ cos\theta_{hrs} \end{pmatrix}$$

Where θ_{hrs} is (obviously) the spectrometer angle.
Now let us express $\hat{q}$ in the transport frame. Defining ψ as the angle between the momentum and its projection on the yz plane and ϕ as the angle between the projection on the yz plane and the z axis (see Fig. 1), if θ is defined as the angle between the momentum and it's projection on the zx plane then $tan\psi = tan\theta cos\phi$, finally, $\vec{q}$ in the transport frame is given by:

$$\vec{q} = \begin{pmatrix} sin\psi \\ cos\psi sin\phi \\ cos\psi cos\phi \end{pmatrix}$$

The transformation matrix can now be formed by using the appropriate column vectors.

2.3 Spin Transport in the HRS

The spin of a particle passing through a magnetic field which components perpendicular to the momentum precesses around the perpendicular components, if we consider the HRS to be a perfect dipole (Fig. 2) than the spin precesses around the transverse magnetic field by an angle χ proportional to the bend angle of the dipole as:

$$\chi = \gamma(\mu - 1)\theta_{bend}$$

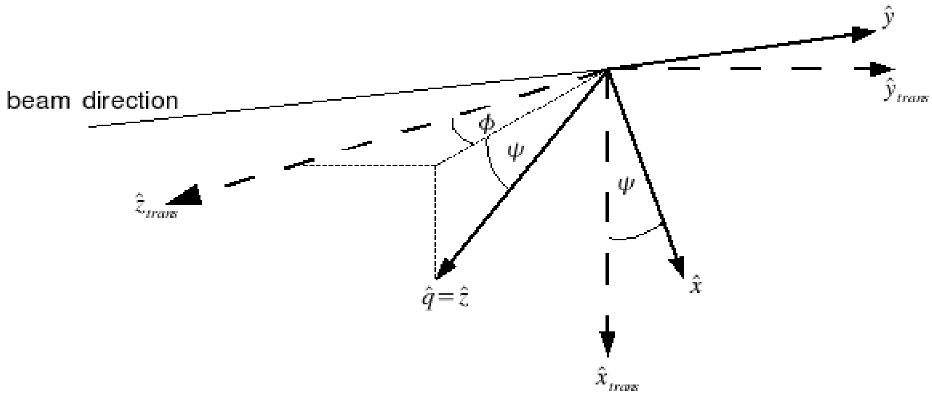

Figure 1: Scattering (Physics) and Transport Frames

Where $\gamma = 1/\sqrt{1-\beta^2}$, θ_p is the bend angle of the momentum trajectory (45° for the HRS dipole in first approximation) and $\mu - 1$ is the anomalous magnetic moment of the particle (1.793 for protons and 0.433 for deuterons).
In the transport frame, therefore, the spin precession matrix is:

$$R_{spin\,transport} = \begin{pmatrix} cos\chi & 0 & sin\chi \\ 0 & 1 & 0 \\ -sin\chi & 0 & cos\chi \end{pmatrix}$$

This is a first order approximation only, the actual field in the HRS is not uni-

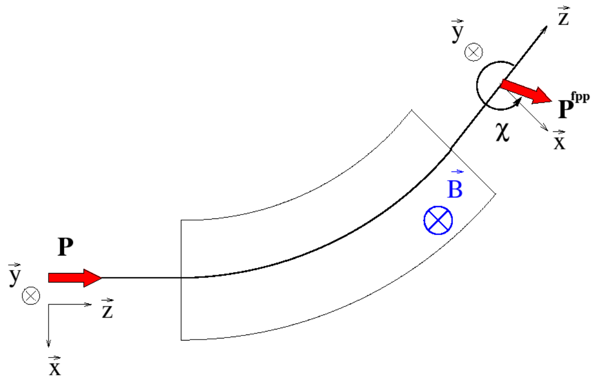

Figure 2: Spin Transport in the HRS

form but distorted in the entrance and exit faces and also contains 3 quadrupole magnets which further destroy the dipole approximation. The full precession matrix has been modeled for the HRS using a COSY, a differential algebra based code [3]. The matrix, however is still a rotation matrix which is applied to the polarization vector in the same way as the dipole approximation matrix.

2.4 The FPP Coordinate System and R_{fpp}

The FPP coordinate frame is defined with the z-axis lying along the incident proton momentum in the HRS, θ_f is the angle between the projection of the incoming track on the xz plane and the z axis and ϕ_f is the angle between the projection on the yz plane and the z (transport) axis. In addition, the angle ψ_f is defined to be angle between the track and the projection on the yz plane (Fig. 3). The relation between the angles can be written as:

$$tan\psi_f = tan\theta_f cos\phi_f$$

Thus the transformation matrix from the transport to the FPP frame can be written as the product of two consecutive rotations, the first by an angle ϕ_f about the $\hat{x}$ axis and the second by an angle ψ_f about the new (rotated) $\hat{y}$ axis:

$$R_{fpp} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\psi_f & sin\psi_f \\ 0 & -sin\psi_f & cos\psi_f \end{pmatrix} \begin{pmatrix} cos\phi_f & 0 & sin\phi_f \\ 0 & 1 & 0 \\ -sin\phi_f & 0 & cos\phi_f \end{pmatrix}$$

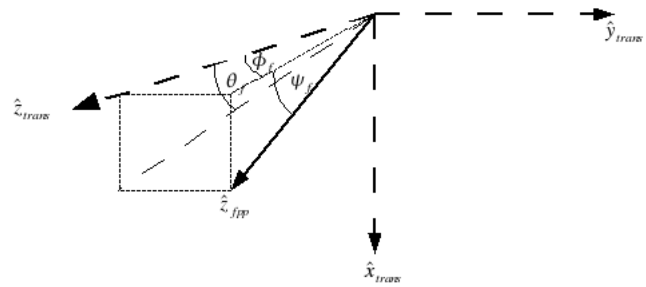

Figure 3: Relation between the FPP and Transport Systems

3 γD Photodisintegration

In the γD reaction the polarization observables are reported in the center of mass frame of the γD system. In that system, $\hat{z}$ is along the proton momentum $\hat{y} = \hat{k} \times \hat{z}$ and $\hat{x} = \hat{y} \times \hat{z}$. In order to transform the observable from the transport frame to the "physics" (CM) frame two consecutive transformations must be performed, a transformation to the hall A frame and a Wigner rotation on the polarization vector.

3.1 The Hall A (lab) Coordinate System

The Hall A coordinate system is defined as $\hat{z}$ along the incident electron beam direction (nominally the direction of the incident photon in the reaction), $\hat{y}$ points vertically upwards and $\hat{x} = \hat{y} \times \hat{z}$ (beam left).
In much the same way as for the elastic case we write the Hall A frame base vectors in the transport frame and use them to construct a transformation matrix. If fact, since in the elastic case $\hat{z} = \hat{q}$ and we get $\hat{y} = \hat{q} \times \hat{k_i}$, other than a change

of sign in the $\hat{x}$ and $\hat{y}$ vectors they are the same ($\hat{k}$ and $\hat{q}$ in the spectrometer frame are unchanged from the elastic case and only the definition of the base vectors changes).

3.2 Wigner Rotation

A well known fact is that the spin direction of a particle is affected by a boost that is not parallel to its momentum, a relativistic effect termed *Wigner rotation* or *Wigner-Thomas precession.* In the γD case since since the physics observables are reported in the CM frame of the γD system while the polarization of the proton is measured (after the appropriate transformation) in the lab frame (which is also the CM frame for the pn pair) one must take into account the effect of the Wigner rotation[4].
In effect the transformation of the proton polarization from the CM frame to the lab frame is as follows:

- The proton is boosted to rest along it's momentum in the CM frame.
- The CM frame is rotated so that the longitudinal components of the proton polarization vector coincides with the z axis of the CM frame.
- A boost is performed along the z axis to bring the CM frame to the lab frame.
- A rotation is performed to align the longitudinal component of the proton polarization vector with the direction of the proton's momentum in the lab frame.
- The proton is boosted along the direction of the momentum (in the lab frame).

The transformation may be written as follows[4]:

$$R[-\Omega_W] = B_z[\beta_{lab}]R_y[\theta_{lab}]B_z[-\beta]R_y[-\theta cm]B_z[-\beta_{cm}]$$

Where $\beta_{cm(lab)}$ is the proton velocity in the CM (lab) frame, $\theta_{cm(lab)}$ if the angle between the proton momentum and the rotated z axis in the CM (lab) frame, β is the relative velocity between the CM and lab frame, R_y represents a rotation about the y axis and B_z is a boost along the z axis. By convention the sign are chosen to make Ω_W positive for $\beta > 0$ (the typical $CM \rightarrow lab$ transformation). By performing the (tedious) matrix multiplication it can be shown that $R[-\Omega]$ is a pure rotation matrix and that spin rotation angle Ω_W is given by:

$$tan\Omega_W = \frac{\beta sin\theta_{cm}}{\gamma_{cm}(\beta cos\theta_{cm} + \beta_{cm})}$$

Where β_{cm} is the proton velocity in the CM frame, β is the velocity of the center of mass with respect to the lab and θ_{cm} is the proton angle with respect to the incident photon in the CM frame. Since the rotation between the two frame is about the $\hat{y}$ axis the transformation matrix is:

$$R_W = \begin{pmatrix} cos\Omega_W & 0 & sin\Omega_W \\ 0 & 1 & 0 \\ -sin\Omega_W & 0 & cos\Omega_W \end{pmatrix}$$

Note that the effect of this rotating is a mixing of the longitudinal and transverse components of the polarization vector.
Thus the complete transformation for the γD reaction is

$$P_{FPP} = R_{fpp} R_{spin\,transport} R_{transport} R_W P_{physics}$$

4 e^4He Scattering

4.1 The Physics Frame and $R_{transport}$

In the e^4He reaction the polarization observables are reported in the scattering plane which is defined in a similar way to the ep scattering plane. Note, however that in this reaction $\hat{q} \neq \hat{p}$ and one must use the original definitions:

$$\hat{z} = \frac{\vec{k_i} - \vec{k_f}}{|\vec{k_i} - \vec{k_f}|} = \hat{q}$$

$$\hat{y} = \frac{\vec{k_i} \times \vec{k_f}}{|\vec{k_i} \times \vec{k_f}|}$$

$$\hat{x} = \hat{y} \times \hat{z}$$

And write the base vectors in an appropriate manner.

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